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A CHARACTERIZATION OF COVERING EQUIVALENCE

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Abstract. Let $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ and $B = \{b_t(\text{mod } m_t)\}_{t=1}^l$ be two systems of residue classes. If $|\{1 \leqslant s \leqslant k : x \equiv a_s \pmod{n_s}\}|$ and $|\{1 \leqslant t \leqslant l : x \equiv b_t \pmod{m_t}\}|$ are equal for all $x \in \mathbb{Z}$, then A and B are said to be covering equivalent. In this paper we characterize the covering equivalence in a simple and new way. Using the characterization we partially confirm a conjecture of R. L. Graham and K. O'Bryant.

1. Introduction

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $a \in \{0, \ldots, n-1\}$, we simply use $a(n)$ to denote the residue class $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \quad (0 \leqslant a_s < n_s) \tag{1.1}$$

of residue classes, the $n_1, \ldots, n_k$ are called its moduli and its *covering function* $w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, \ldots\}$ is given by

$$w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|. \tag{1.2}$$

(The covering function $w_\emptyset$ of an empty system is regarded as the zero function.) The periodic function $w_A(x)$ has many surprising properties (cf. [S03a], [S04] and [S05a]).

Let m be a positive integer. If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then (1.1) is said to be an *exact m-cover* of $\mathbb{Z}$ as in [S95] and [S96]. Recently Z. W. Sun (cf. [S04] and [S05b]) showed that (1.1) forms an exact m-cover of $\mathbb{Z}$ if it covers $|S(n_1, \ldots, n_k)|$ consecutive integers exactly m times, where

$$S(n_1, \ldots, n_k) = \left\{ \frac{r}{n_s} : \ r = 0, \ldots, n_s - 1;\ s = 1, \ldots, k \right\}. \tag{1.3}$$

For problems and results on covers of $\mathbb{Z}$ by residue classes, the reader is referred to [FFKPY], [G04] and [S03b].

For two finite systems $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$, Sun [S89] called A and B *covering equivalent* (in short, $A \sim B$) if they have the same

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covering function (i.e., $w_A = w_B$). Thus (1.1) is an exact m-cover of $\mathbb{Z}$ if and only if (1.1) is covering equivalent to the system consisting of m copies of $0(1)$.

In [S01] and [S02] Sun characterized the covering equivalence by various systems of equalities. In this paper we present a simple characterization involving roots of unity. Namely, we have the following result.

Theorem 1.1. *Let $A = \{a_s(n_s)\}_{s=1}^k$ $(0 \leqslant a_s < n_s)$ and $B = \{b_t(m_t)\}_{t=1}^l$ $(0 \leqslant b_t < m_t)$ be two finite systems of residue classes. Let p be a prime greater than $|S(n_1, \dots, n_k, m_1, \dots, m_l)|$, and let ζ_p be a primitive pth root of unity. Then A and B are covering equivalent if and only if*

$$\sum_{s=1}^{k} \frac{\zeta_p^{a_s}}{1-\zeta_p^{n_s}} = \sum_{t=1}^{l} \frac{\zeta_p^{b_t}}{1-\zeta_p^{m_t}}. \tag{1.4}$$

Corollary 1.1. (1.1) *forms an exact m-cover of $\mathbb{Z}$ if and only if*

$$\sum_{s=1}^{k} \frac{e^{2\pi i a_s/p}}{1-e^{2\pi i n_s/p}} = \frac{m}{1-e^{2\pi i/p}}, \tag{1.5}$$

where p is any fixed prime greater than $|S(n_1, \dots, n_k)|$.

Proof. Simply apply Theorem 1.1 with B consisting m copies of $0(1)$. $\square$

Remark 1.1. In 1975 Š. Znám [Z75a] used the transcendence of e to prove that (1.1) is a disjoint cover (i.e., exact 1-cover) of $\mathbb{Z}$ if and only if

$$\sum_{s=1}^{k} \frac{e^{a_s}}{1-e^{n_s}} = \frac{1}{1-e}.$$

Corollary 1.2. *Suppose that for nonempty system* (1.1) *we have*

$$\sum_{s=1}^{k} \frac{e^{2\pi i a_s/p}}{1-e^{2\pi i n_s/p}} = 0$$

where p is a prime. Then

$$n_1 + \dots + n_k - k + 1 \geqslant |S(n_1, \dots, n_k)| \geqslant p. \tag{1.6}$$

Proof. Clearly $|S(n_1, \dots, n_k)| \leqslant n_1 + \dots + n_k - k + 1$. Since we don't have $A \sim \emptyset$, applying Theorem 1.1 with $B = \emptyset$ we find that $|S(n_1, \dots, n_k)|$ cannot be smaller than p. This concludes the proof. $\square$

Corollary 1.2 partially confirms the following conjecture arising from the study of Fraenkel's conjecture on disjoint covers of $\mathbb{N}$ by Beatty sequences.

Graham–O'Bryant Conjecture ([GO]). *Let $n_1, \dots, n_k$ be distinct positive integers less than and relatively prime to $q \in \mathbb{Z}^+$. If $a_1, \dots, a_k \in \mathbb{Z}$ and*

$$\sum_{s=1}^{k} \frac{e^{2\pi i a_s/q}}{1 - e^{2\pi i n_s/q}} = 0,$$

then we must have $\sum_{s=1}^{k} n_s \geqslant q$.

The following example shows that we cannot replace the prime p in Corollary 1.2 or Theorem 1.1 by a composite number.

Example 1.1. Let $q > 1$ be an integer and let p be a prime divisor of q. Then, for any $n = 1, \dots, q-1$, we have

$$\sum_{s=0}^{p-1} \frac{e^{2\pi i (sq/p)/q}}{1 - e^{2\pi i n/q}} = \frac{\sum_{s=0}^{p-1} e^{2\pi i s/p}}{1 - e^{2\pi i n/q}} = 0$$

but $|S(n, \dots, n)| = n < q$. Thus the conditions $0 \leqslant a_s < n_s$ $(s = 1, \dots, k)$ in Corollary 1.2 cannot be cancelled. If q is composite, then there are $q/p - 1 > 0$ integers in the interval $((p-1)q/p,\ q-1]$. So we cannot substitute a composite number for the prime p in Corollary 1.2.

Corollary 1.3. *Let $A = \{a_s(n_s)\}_{s=1}^{k}$ $(0 \leqslant a_s < n_s)$ and $B = \{b_t(m_t)\}_{t=1}^{l}$ $(0 \leqslant b_t < m_t)$ both have distinct moduli. Let p be a prime greater than $|S(n_1, \dots, n_k, m_1, \dots, m_l)|$, and let ζ_p be a primitive pth root of unity. Then A and B are identical if and only if* (1.4) *holds.*

Proof. By a result of Znám [S75b], A and B are identical if they have the same covering function. Combining this with Theorem 1.1 we immediately get the desired result. □

Observe that $A = \{a_s(n_s)\}_{s=1}^{k}$ and $B = \{b_t(m_t)\}_{t=1}^{l}$ are covering equivalent if and only if

$$\sum_{\substack{s=1 \\ x \in a_s(n_s)}}^{k} 1 + \sum_{\substack{t=1 \\ x \in b_t(m_t)}}^{l} (-1) = 0 \quad \text{for every } x \in \mathbb{Z}.$$

Thus Theorem 1.1 has the following equivalent form which will be proved in the next section.

Theorem 1.2. *Let $\mathcal{A} = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^{k}$ where $\lambda_s, a_s, n_s \in \mathbb{Z}$ and $0 \leqslant a_s < n_s$. Let $p > |S(n_1, \dots, n_k)|$ be a prime, and let ζ_p be any primitive pth root of unity. Then $\mathcal{A} \sim \emptyset$ (i.e., $w_{\mathcal{A}}(x) = \sum_{1 \leqslant s \leqslant k,\, x \in a_s(n_s)} \lambda_s = 0$ for all $x \in \mathbb{Z}$) if and only if*

$$\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0. \tag{1.7}$$

2. Proof of Theorem 1.2

Let $S = S(n_1, \ldots, n_k)$. As $p > |S| \geqslant \max\{n_1, \ldots, n_k\}$, there is a common multiple $N \in \mathbb{Z}^+$ of the moduli $n_1, \ldots, n_k$ such that $N \equiv 1 \pmod p$. Just as in [S05a], we have

$$\begin{aligned}\sum_{r=0}^{N-1} w_{\mathcal{A}}(r)z^r =& \sum_{r=0}^{N-1} \sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid a_s-r}} \lambda_s z^r = \sum_{s=1}^{k} \lambda_s \sum_{\substack{0\leqslant r<N\\ r\in a_s(n_s)}} z^r\\ =&\sum_{s=1}^{k} \lambda_s z^{a_s} \sum_{0\leqslant q<N/n_s} (z^{n_s})^q\\ =&N \sum_{\substack{1\leqslant s\leqslant k\\ z^{n_s}=1}} \frac{\lambda_s}{n_s} z^{a_s} + (1-z^N) \sum_{\substack{1\leqslant s\leqslant k\\ z^{n_s}\neq 1}} \lambda_s \frac{z^{a_s}}{1-z^{n_s}}.\end{aligned}$$

Thus

$$\sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = (1-\zeta_p^N) \sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1-\zeta_p^{n_s}}.$$

It follows that

$$\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1-\zeta_p^{n_s}} = 0 \iff \sum_{l=0}^{p-1} c_l \zeta_p^l = 0, \tag{2.1}$$

where

$$c_l = \sum_{\substack{x=0\\ x\in l(p)}}^{N-1} w_{\mathcal{A}}(x) \in \mathbb{Z}.$$

If $w_{\mathcal{A}}(x) = 0$ for all $x \in \mathbb{Z}$, then (1.7) holds by the above.

Below we assume (1.7). Then $\sum_{l=0}^{p-1} c_l\zeta_p^l = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = 0$. In the case $N = 1$, it follows that $w_{\mathcal{A}}(x) = w_{\mathcal{A}}(0) = 0$ for all $x \in \mathbb{Z}$.

Now suppose $N > 1$. Clearly $N > p$ as $N \equiv 1 \pmod p$. Since $1 + x + \cdots + x^{p-1} = (x^p - 1)/(x - 1)$ is the minimal polynomial of ζ_p over the field of rational numbers, we must have $c_0 = c_1 = \cdots = c_{p-1}$. (See also M. Newman [N71].) Observe that if $x \in \mathbb{Z}$ then

$$w_{\mathcal{A}}(x) = \sum_{s=1}^{k} \frac{\lambda_s}{n_s} \sum_{r=0}^{n_s-1} e^{2\pi i \frac{a_s-x}{n_s} r} = \sum_{\alpha\in S} e^{-2\pi i\alpha x} \sum_{\substack{s=1\\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s}. \tag{2.2}$$

(This trick appeared in [S91] and [S04].) Since $|S| < p$, for each $l = 0, \ldots, |S|$ we have

$$c_l = \sum_{\substack{x=0 \\ x\in l(p)}}^{N-1} w_{\mathcal{A}}(x) = \sum_{\alpha\in S} \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} \sum_{\substack{x=0 \\ x\in l(p)}}^{N-1} e^{-2\pi i\alpha x}$$
$$= \sum_{\alpha\in S} e^{-2\pi i\alpha l} \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} \sum_{j=0}^{\lfloor (N-1-l)/p\rfloor} e^{-2\pi i\alpha pj},$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. If $l \in \{1, \ldots, |S|\}$ then

$$\left\lfloor \frac{N-1-l}{p} \right\rfloor = \frac{N-1}{p} + \left\lfloor \frac{-l}{p} \right\rfloor = \frac{N-1}{p} - 1;$$

if $\alpha \in S \setminus \{0\}$ then

$$C(\alpha) := \sum_{j=0}^{(N-1)/p-1} e^{-2\pi i\alpha pj} = \frac{1-(e^{-2\pi i\alpha p})^{(N-1)/p}}{1-e^{-2\pi i\alpha p}} = \frac{1-e^{2\pi i\alpha}}{1-e^{-2\pi i\alpha p}} \neq 0.$$

Let $c = c_0 = \cdots = c_{p-1}$. By the above,

$$\sum_{\alpha\in S} e^{-2\pi i\alpha j} f(\alpha) = c$$

for every $j = 0, \ldots, |S|-1$, where

$$f(0) = \frac{N-1}{p} \sum_{s=1}^{k} \frac{\lambda_s}{n_s}$$

and

$$f(\alpha) = e^{-2\pi i\alpha} C(\alpha) \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} \quad \text{for } \alpha \in S \setminus \{0\}.$$

Let $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{|S|-1}$ be all the distinct elements of S. Now that

$$\sum_{t=0}^{|S|-1} e^{-2\pi i\alpha_t j} f(\alpha_t) = c \quad \text{for each } j = 0, \ldots, |S|-1,$$

by Cramer's rule $D_t = Df(\alpha_t)$ vanishes for every $t = 1, \ldots, |S|-1$, where $D = \det((e^{-2\pi i\alpha_t})^j)_{0\leqslant j,t<|S|}$ is of Vandermonde's type and hence nonzero. Therefore

$$\sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} = 0 \quad \text{for all } \alpha \in S \setminus \{0\}$$

and hence $w_{\mathcal{A}}(x) = \sum_{s=1}^{k} \lambda_s/n_s$ for all $x \in \mathbb{Z}$ by (2.2). It follows that

$$0 = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = \sum_{s=1}^{k} \frac{\lambda_s}{n_s}(1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{N-1}) = \sum_{s=1}^{k} \frac{\lambda_s}{n_s} \cdot \frac{1-\zeta_p^N}{1-\zeta_p}.$$

So $\sum_{s=1}^{k} \lambda_s/n_s = 0$ and hence $\mathcal{A} \sim \emptyset$. We are done.

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